South East Asian J. of Mathematics and Mathematical Sciences
Vol. 20, No. 3 (2024), pp. 247-256

DOI: 10.56827/SEAJMMS.2024.2003.18 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

ARITHMETIC CONTINUITY IN QUASI CONE METRIC SPACES

Iqbal Kour, Sahil Billawria and Shallu Sharma

Department of Mathematics,
University of Jammu, Jammu, INDIA

E-mail : iqbalkour208@gmail.com, sahilbillawria2@gmail.com,
shallujamwal09@gmail.com

(**Received:** *Dec. 13, 2023* **Accepted:** *Nov. 22, 2024* **Published:** *Dec. 30, 2024*)

Abstract: The current study defines and examines the concept of arithmetic continuity in quasi-cone metric spaces. This work introduces several new concepts, including forward and backward arithmetic convergence, arithmetic ff-continuity, fb-continuity, forward and backward arithmetic compactness, and uniform continuity. We have determined the conditions under which the uniform limit of an arithmetic ff-continuous function is again an arithmetic ff-continuous function. In quasi cone metric spaces, certain arithmetic compactness results are also proved. We have also proved some interesting results pertaining to these concepts.

Keywords and Phrases: Arithmetic continuity, arithmetic convergence, arithmetic compactness.

2020 Mathematics Subject Classification: 40A35 40A05, 26A15.

1. Introduction

If symmetric condition is eliminated from the definition of metric (see [2, 5, 12, 13, 16]) then the distance function is said to be quasi metric. Quasi metric has a variety of applications in pure and applied mathematics, as well as material science (see [4]). Various definitions of quasi cone metric have been given by various authors (see, for example, [1]). Since then, much study has been conducted on the quasi cone metric, particularly on fixed point theory (see, for example [2]). The notion of arithmetic convergence was introduced by Ruckle [14] in the form of a sequence $\{\mathfrak{x}_n\}$ defined on the set of natural numbers. The sequence $\{\mathfrak{x}_n\}$ is said to be arithmetic convergent if for every $\mathfrak{u} > 0$ there exists $m \in \mathbb{Z}$ such that for each $n \in \mathbb{Z}, |\mathfrak{x}_n - \mathfrak{x}_{<n,m>}| < \mathfrak{u}$, where the greatest common divisor of n and m is denoted as $< n, m >$. Another definition of arithmetic convergence given by Cakalli in [3] as: a sequence $\{\mathfrak{x}_n\}$ is called arithmetically convergent if for each $u > 0$ there exists $m_0 \in \mathbb{Z}$ such that for all $n, m \in \mathbb{Z}$ that satisfy $< n, m > \geq m_0$, we have $|\mathfrak{x}_n - \mathfrak{x}_{<n,m>}| < u$. For comprehensive details on arithmetic continuity, arithmetic convergence, ideal convergent sequences and related notions one can refer [3, 6, 7, 8, 10, 11, 17, 18, 21, 22]. In this paper, the concepts of forward and backward arithmetic convergence in a quasi cone metric space are firstly introduced. These concepts are used to define forward and backward arithmetic continuity in quasi cone metric spaces which are further utilized to obtain some fascinating results. The notion of forward and backward arithmetic compactness is also introduced.

2. Preliminaries

Throughout this paper, the partial order relation is denoted by $\leq$.

Definition 2.1. [9] *Let $\mathfrak{E} \subseteq \mathbb{R}$ be a Banach space. A set $\mathfrak{P}$ contained in $\mathfrak{E}$ is said to be a cone if*

(i) $\mathfrak{P}$ is non-empty, closed and non-zero.

(ii) $\mathfrak{x}, \mathfrak{y}$ are elements of $\mathfrak{P}$ and $\mathfrak{s}, \mathfrak{t} \in \mathbb{R}_{\geq 0}$, then $\mathfrak{s}\mathfrak{x} + \mathfrak{t}\mathfrak{y}$ is an element of $\mathfrak{P}$.

(iii) the intersection of $\mathfrak{P}$ and $-\mathfrak{P}$ is $\{0\}$.

Throughout this paper we shall assume that $\mathfrak{P}$ is a cone in a Banach space such that $int(\mathfrak{P})$ denotes interior of $\mathfrak{P}$. Also, we shall denote closure of any set $\mathfrak{B}$ as $Cl(\mathfrak{B})$. A partial ordering relation $\leq$ on $\mathfrak{E}$ in relation to $\mathfrak{P}$ is defined as: $\mathfrak{x} \ll \mathfrak{y}$ if and only if $\mathfrak{y} - \mathfrak{x}$ is an element of int$\mathfrak{P}, \mathfrak{x} \leq \mathfrak{y}$ if and only if $\mathfrak{y} - \mathfrak{x}$ is an element of $\mathfrak{P}$. The partial ordering $\mathfrak{x} < \mathfrak{y}$ symbolizes $\mathfrak{x} \leq \mathfrak{y}$, $\mathfrak{x} \neq \mathfrak{y}$.

Definition 2.2. [1] *Let $\mathfrak{X}$ be a non-empty set and define a mapping $d : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{E}$ as follows:*

(i) $d(\mathfrak{x}, \mathfrak{y}) \geq 0$ for $\mathfrak{x}, \mathfrak{y} \in \mathfrak{X}$.

(ii) $d(\mathfrak{x}, \mathfrak{y}) = 0$ if and only if $\mathfrak{x} = \mathfrak{y}$.

(iii) $d(\mathfrak{x}, \mathfrak{y}) \leq d(\mathfrak{x}, \mathfrak{z}) + d(\mathfrak{z}, \mathfrak{y})$ for $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathfrak{X}$.

Then $(\mathfrak{X}, d)$ is said to be a quasi cone metric space.

Further, we shall assume that $(\mathfrak{X}, d)$ is a quasi cone metric space.

Definition 2.3. [15] *Let $\{\mathfrak{x}_n\}$ be a sequence in $\mathfrak{X}$. The sequence $\{\mathfrak{x}_n\}$ is said to be* *forward convergent (resp. backward convergent) to* $\mathfrak{x}_0$ *in* $\mathfrak{X}$ *if for each* $\mathfrak{u} \gg 0, \mathfrak{u} \in \mathfrak{E}$ *there exists* $\mathfrak{N} \in \mathbb{N}$ *so that for every* $\mathfrak{n} \geq \mathfrak{N}$*, we have* $d(\mathfrak{x}_0, \mathfrak{x}_n) \ll u$ *(resp.* $d(\mathfrak{x}_n, \mathfrak{x}_0) \ll \mathfrak{u}$*). We denote it as* $\mathfrak{x}_n \xrightarrow{f} \mathfrak{x}_0$ *(resp.* $\mathfrak{x}_n \xrightarrow{b} \mathfrak{x}_0$*).*

Definition 2.4. [15] *Let* $\{\mathfrak{x}_n\}$ *be a sequence in* $(\mathfrak{X}, d)$*. Then the sequence* $\{\mathfrak{x}_n\}$ *is said to be forward Cauchy (resp. backward Cauchy) if for each* $\mathfrak{u} \gg 0, \mathfrak{u} \in \mathfrak{E}$ *there exists* $\mathfrak{N} \in \mathbb{N}$ *such that for each* $m \geq n \geq \mathfrak{N}, d(\mathfrak{x}_n, \mathfrak{x}_m) \ll \mathfrak{u}$ *(resp.* $d(\mathfrak{x}_m, \mathfrak{x}_n) \ll \mathfrak{u}$*).*

Definition 2.5. [19] $(\mathfrak{X}, d)$ *is said to be forward sequentially compact if each sequence in* $\mathfrak{X}$ *has a forward convergent subsequence.*

3. Main Results

Definition 3.1. *Let* $\{\mathfrak{x}_n\}$ *be a sequence in* $\mathfrak{X}$*. Then* $\{\mathfrak{x}_n\}$ *is said to be forward arithmetic convergent (resp. backward arithmetic convergent) if for every* $\mathfrak{u} \in \mathfrak{E}, u \gg 0$ *there exists* $\mathfrak{N} \in \mathbb{Z}$ *such that* $d(\mathfrak{x}_{<n,m>}, \mathfrak{x}_n) \ll u$*(resp.* $d(\mathfrak{x}_n, \mathfrak{x}_{<n,m>}) \ll u$*), whenever* $< n, m > \geq \mathfrak{N}, n, m \in \mathbb{Z}$*. It shall be denoted as* $\mathfrak{x}_n \overset{amf}{\to} \mathfrak{x}_{<n,m>}$ *(resp.* $\mathfrak{x}_n \overset{amb}{\to} \mathfrak{x}_{<n,m>}$*).*

Hereafter, we assume that $(\mathfrak{X}, d_\mathfrak{X})$ and $(\mathfrak{Y}, d_\mathfrak{Y})$ are two quasi cone metric spaces.

Definition 3.2. *A function* f *from* $\mathfrak{X}$ *to* $\mathfrak{Y}$ *is said to be arithmetic* $ff-$*continuous (resp. arithmetic* $fb-$*continuous) at a point* $\mathfrak{x} \in \mathfrak{X}$ *if* $\mathfrak{x}_n \overset{amf}{\to} \mathfrak{x}$ *in* $(\mathfrak{X}, d_\mathfrak{X})$ *implies that* $f(\mathfrak{x}_n) \overset{amf}{\to} f(\mathfrak{x})$*(resp.* $f(\mathfrak{x}_n) \overset{amb}{\to} f(\mathfrak{x})$*) in* $(\mathfrak{Y}, d_\mathfrak{Y})$*.*

Definition 3.3. *A function* f *from* $\mathfrak{X}$ *to* $\mathfrak{Y}$ *is said to be uniformly continuous if for each* $\mathfrak{x}, \mathfrak{y} \in \mathfrak{X}, u' \in \mathfrak{E}', u' \gg 0$ *there exists* $u \in \mathfrak{E}, u \gg 0$ *if* $d_\mathfrak{X}(\mathfrak{x}, \mathfrak{y}) \ll u$ *implies that* $d_\mathfrak{Y}(f(\mathfrak{x}), f(\mathfrak{y})) \ll u'$*.*

Note that forward uniform continuity and backward uniform continuity are same.

Theorem 3.4. *A function* f *from* $\mathfrak{X}$ *to* $\mathfrak{Y}$ *is arithmetic* $ff-$*continuous if* f *is uniformly continuous.*

Proof. Suppose that f is uniformly continuous and $\{\mathfrak{x}_n\}$ is a sequence in $\mathfrak{X}$ that is forward arithmetic convergent. Then for every $u' \in \mathfrak{E}', u' \gg 0$ there exists $u \in \mathfrak{E}, u \gg 0$ such that $d_\mathfrak{Y}(f(\mathfrak{x}), f(\mathfrak{y})) \ll u'$, whenever $d_\mathfrak{X}(\mathfrak{x}, \mathfrak{y}) \ll u$ as f is uniformly continuous. Also, $\{\mathfrak{x}_n\}$ is forward arithmetic convergent in $\mathfrak{X}$. Then for above $u \gg 0$, there exists $n_0 \in \mathbb{N}$ such that for all $m, n \in \mathbb{Z}$ that satisfy $< n, m > \geq n_0$, we have $d_\mathfrak{Y}(f(\mathfrak{x}_{<n,m>}), f(\mathfrak{x}_n)) \ll u'$ when $d_\mathfrak{X}(\mathfrak{x}_{<n,m>}, \mathfrak{x}_n) \ll u$ for each n. Therefore, $\{f(\mathfrak{x}_n)\}$ is forward arithmetic convergent sequence. Hence f is arithmetic $ff-$continuous.

Definition 3.5. *A sequence* $\{f_n\}$ *from* $\mathfrak{X}$ *to* $\mathfrak{Y}$ *is said to be forward arithmetic convergent (resp. backward arithmetic convergent) if for every* $u' \in \mathfrak{E}', u' \gg 0$

and for every $\mathfrak{x} \in \mathfrak{X}$ *there exists* $n_0 \in \mathbb{N}$ *such that for each* $m, n \in \mathbb{Z}$ *that satisfy* $< m, n > \geq n_0$, *we have* $d_{\mathfrak{Y}}(f_{<n,m>}(\mathfrak{x}), f_n(\mathfrak{x})) \ll u'$ *(resp.* $d_{\mathfrak{Y}}(f_n(\mathfrak{x}), f_{<n,m>}(\mathfrak{x})) \ll u'$*).*

Theorem 3.6. *Let* $(\mathfrak{X}, d_{\mathfrak{X}})$ *and* $(\mathfrak{Y}, d_{\mathfrak{Y}})$ *be two quasi cone metric spaces. If* $\{f_n\}$ *is a sequence of forward arithmetic convergent functions from* $\mathfrak{X}$ *to* $\mathfrak{Y}$ *and* $\mathfrak{x}_0 \in \mathfrak{X}$ *so that* $f_n(\mathfrak{x})$ *tends to* $\mathfrak{y}_n$ *as* $\mathfrak{x}$ *tends to* $\mathfrak{x}_0$, *then the sequence* $\{\mathfrak{y}_n\}$ *is also forward arithmetic convergent.*

Proof. By the definition of forward arithmetic convergence, for $u' \in \mathfrak{E}', u' \gg 0$ there exists $n_0 \in \mathbb{N}$ such that for every $m, n \in \mathbb{Z}$ that satisfy $< n, m > \geq n_0$ and for each $x \in \mathfrak{X}$, we have $d_{\mathsf{Y}}(f_{<n,m>}(x), f_n(x)) \ll u'$. Fix n, m and suppose $\mathfrak{x}$ tends to $\mathfrak{x}_0$, we have $d_{\mathfrak{Y}}(\mathfrak{y}_{<n,m>}, \mathfrak{y}_n) \ll u'$. Therefore, $\{\mathfrak{y}_n\}$ is forward arithmetic convergent. This proves the theorem.

Theorem 3.7. *If* $\{f_n\}$ *is a sequence of backward arithmetic convergent functions from* $\mathfrak{X}$ *to* $\mathfrak{Y}$ *and* $\mathfrak{x}_0 \in \mathfrak{X}$ *so that* $f_n(\mathfrak{x})$ *tends to* $\mathfrak{y}_n$ *as* $\mathfrak{x}$ *tends to* $\mathfrak{x}_0$, *then the sequence* $\{\mathfrak{y}_n\}$ *is also backward arithmetic convergent.*

Proof. The proof is omitted as it follows trivially from Theorem 3.6.

Theorem 3.8. *If* $\{f_n\}$ *is a sequence of arithmetic* $ff-$*continuous functions from* $\mathfrak{X}$ *to* $\mathfrak{Y}$, *forward convergence is equivalent to backward convergence in* $\mathfrak{Y}$ *and* $\{f_n\}$ *is forward convergent to* f *uniformly. Then* f *is arithmetic* $ff-$*continuous.*

Proof. Choose $u' \in \mathfrak{E}', u' \gg 0$. Suppose that $\{\mathfrak{x}_n\}$ is a forward arithmetic convergent sequence in $\mathfrak{X}$. Since $\{f_n\}$ is uniformly forward convergent to f, there exists $\mathfrak{N} \in \mathbb{N}$ such that $d_{\mathfrak{Y}}(f(\mathfrak{x}), f_n(\mathfrak{x})) \ll \frac{u'}{3}$ for all $n \geq \mathfrak{N}$ and $x \in \mathfrak{X}$. Particularly, $f_n(\mathfrak{x}_{<n,m>})$ is forward convergent to $f(\mathfrak{x}_{<n,m>})$ and hence $f_n(\mathfrak{x}_{<n,m>})$ is backward convergent to $f(\mathfrak{x}_{<n,m>})$. Therefore, there exist $\mathfrak{M} \in \mathbb{N}$ such that $d_{\mathsf{Y}}(f_n(\mathfrak{x}_{<n,m>}), f(\mathfrak{x}_{<n,m>})) \ll \frac{u'}{3}$ for all $n \geq \mathfrak{M}$. Let $\mathfrak{N}' = \max.(\mathfrak{N}, \mathfrak{M})$. Also, $\{f_n\}$ is a sequence of arithmetic $ff-$continuous functions. Particularly, $f_{\mathfrak{N}'}$ is arithmetic $ff-$continuous function. Since forward convergence and backward convergence in $\mathfrak{Y}$ are equivalent. Then $f_{\mathfrak{N}'}$ is also arithmetic $fb-$continuous. Therefore, there exists $n_0 > \mathsf{N}$ and $u \in \mathfrak{E}, u \gg 0$ such that $d_{\mathfrak{X}}(\mathfrak{x}_{<n,m>}, \mathfrak{x}_n) \ll u$ implies that $d_{\mathfrak{Y}}(f_{\mathfrak{N}'}(\mathfrak{x}_n), f_{\mathfrak{N}'}(\mathfrak{x}_{<n,m>}) \ll \frac{u'}{3} \forall m, n \in \mathbb{N}$ such that $< n, m > \geq m_0$. Moreover, for $d_{\mathfrak{X}}(x_{<n,m>}, x_n) \ll u$ and $< n, m > \geq m_0$, we see that

$$\begin{aligned} d_{\mathfrak{Y}}(f(\mathfrak{x}_n), f(\mathfrak{x}_{<n,m>})) &\leq d_{\mathfrak{Y}}(f(\mathfrak{x}_n), f_{\mathfrak{N}'}(\mathfrak{x}_n)) \\ &+ d_{\mathfrak{Y}}(f_{\mathfrak{N}'}(\mathfrak{x}_n), f_{\mathfrak{N}'}(\mathfrak{x}_{<n,m>})) + d_{\mathfrak{Y}}(f_{\mathfrak{N}'}(\mathfrak{x}_{<n,m>}), f(\mathfrak{x}_{<n,m>})) \\ &\ll \frac{u'}{3} + \frac{u'}{3} + \frac{u'}{3} \\ &= u'. \end{aligned}$$

Hence f is arithmetic $fb-$continuous. As forward convergence is equivalent to backward convergence f is also arithmetic $ff-$continuous. This proves the theorem.

Theorem 3.9. *Let forward convergence in $\mathfrak{Y}$ is equivalent to backward convergence. Then The set of all arithmetic $ff-$continuous functions from $\mathfrak{X}$ to $\mathfrak{Y}$ is a closed subset of all continuous functions from $\mathfrak{X}$ to $\mathfrak{Y}$. That is, $Cl(\mathfrak{A}_{mff})(\mathfrak{X},\mathfrak{Y}) = \mathfrak{A}_{mff}(\mathfrak{X},\mathfrak{Y})$. Here $\mathfrak{A}_{mff}(\mathfrak{X},\mathfrak{Y})$ is the set of all functions from $\mathfrak{X}$ to $\mathfrak{Y}$ that are arithmetic $ff-$continuous.*

Proof. Suppose $f \in Cl(\mathfrak{A}_{mff})(\mathfrak{X},\mathfrak{Y})$. Then there is a sequence $\{f_n\}$ in $\mathfrak{A}_{mff}(\mathfrak{X},\mathfrak{Y})$ such that $\{f_n\}$ is forward convergent to f. Let $u' \in \mathfrak{E}', u' \gg 0$. Consider a forward arithmetic convergent sequence $\{\mathfrak{x}_n\}$ in $\mathfrak{X}$. As f_n is forward convergent uniformly to f. Then for all $\mathfrak{x} \in \mathfrak{X}$ there exists $\mathfrak{N}_1 \in \mathbb{N}$ such that $d_{\mathfrak{Y}}(f(\mathfrak{x}), f_n(\mathfrak{x}) \ll \frac{u'}{3}, \forall n \geq \mathfrak{N}_1$. Particularly, $f_n(\mathfrak{x}_{<n,m>})$ is forward convergent to f. Also, $f_n(\mathfrak{x}_{<n,m>})$ is backward convergent to f as forward convergence is equivalent to backward convergence. Hence there exists $\mathfrak{N}_2 \in \mathbb{N}$ such that $d_{\mathfrak{Y}}(f_n(\mathfrak{x}_{<n,m>}), f(\mathfrak{x}_{<n,m>})) \ll \frac{u'}{3} \forall n \geq \mathfrak{N}_2$. Choose $\mathfrak{N} = \max.\{\mathfrak{N}_1, \mathfrak{N}_2\}$. Also, $\{f_n\}$ is arithmetic $ff-$continuous. Particularly, $f_{\mathfrak{N}}$ is arithmetic $ff-$continuous. Since forward convergence in $\mathfrak{Y}$ is equivalent to backward convergence we see that $f_{\mathfrak{N}}$ is also arithmetic $fb-$continuous. Hence there exists $m_0 > \mathfrak{N}$ and $u \in \mathfrak{E}, u \gg 0$ such that $d_{\mathfrak{X}}(\mathfrak{x}_{<n,m>}, \mathfrak{x}_n) \ll u$ implies that $d_{\mathfrak{Y}}(f_{\mathfrak{N}}(\mathfrak{x}_n), f_{\mathfrak{N}}(\mathfrak{x}_{<n,m>})) \ll \frac{u'}{3}$, $\forall\ m, n \in \mathbb{Z}$ and $< m, n > \geq m_0$. Therefore, for $d_{\mathfrak{X}}(\mathfrak{x}_{<n,m>}, \mathfrak{x}_n) \ll \mathfrak{u}$ and $< m, n > \geq m_0$, we get

$$\begin{aligned} d_{\mathfrak{Y}}(f(\mathfrak{x}_n), f(\mathfrak{x}_{<n,m>})) &\leq d_{\mathfrak{Y}}(f(\mathfrak{x}_n), f_{\mathfrak{N}}(\mathfrak{x}_n)) \\ &+ d_{\mathfrak{Y}}(f_{\mathfrak{N}}(\mathfrak{x}_n), f_{\mathfrak{N}}(\mathfrak{x}_{<n,m>})) + d_{\mathfrak{Y}}(f_{\mathfrak{N}}(\mathfrak{x}_{<n,m>}), f(\mathfrak{x}_{<n,m>})) \\ &\ll \frac{u'}{3} + \frac{u'}{3} + \frac{u'}{3} \\ &= u'. \end{aligned}$$

Hence f is arithmetic $fb-$continuous. Also f is arithmetic $ff-$continuous as forward convergence is equivalent to backward convergence. Therefore, $\mathfrak{A}_{mff}(\mathfrak{X},\mathfrak{Y})$ contains f. Hence the proof.

Definition 3.10. *A function $f : \mathfrak{X} \to \mathfrak{Y}$ is said to be forward $(c\mathfrak{A}\mathfrak{C})-$continuous if it maps forward convergent sequences in $\mathfrak{X}$ to forward arithmetic convergent sequences in $\mathfrak{Y}$. That is, $\{f(\mathfrak{x}_n)\}$ is a forward arithmetic convergent sequence in $\mathfrak{Y}$, whenever $\{\mathfrak{x}_n\}$ is a forward convergent sequence in $\mathfrak{X}$.*

Theorem 3.11. *Let forward convergence in $\mathfrak{Y}$ is equivalent to backward convergence and $\{f_n\}$ be a sequence consisting of forward $(c\mathfrak{A}\mathfrak{C})-$continuous functions from $\mathfrak{X}$ to $\mathfrak{Y}$ such that $\{f_n\}$ is forward convergent to f uniformly. Then f is forward $(c\mathfrak{A}\mathfrak{C})-$continuous.*

Proof. Let $u' \in \mathfrak{E}', u' \gg 0$ and consider a sequence $\{\mathfrak{x}_n\}$ in $\mathfrak{X}$ that is forward convergent. As $\{f_n\}$ is forward convergent to f uniformly, then there exists $\mathfrak{N}_1 \in \mathbb{N}$ such that $d_{\mathfrak{Y}}(f(x), f_n(x)) \ll \frac{u'}{3}$, $\forall\ \mathfrak{x} \in \mathfrak{X}$ and $n \geq \mathfrak{N}_1$. Particularly, $\{f_n(\mathfrak{x}_n)\}$ is forward convergent to f. Since forward convergence is equivalent to backward convergence. Then $\{f_n(\mathfrak{x}_n)\}$ is backward convergent to f. Hence there exists $\mathfrak{N}_2 \in \mathbb{N}$ such that $d_{\mathfrak{Y}}(f_n(\mathfrak{x}_n), f(\mathfrak{x}_n)) \ll \frac{u'}{3}$, $\forall\ n \geq \mathfrak{N}_2$. Take $\mathfrak{N} = \max\{\mathfrak{N}_1, \mathfrak{N}_2\}$. Since f_n is forward $(c\mathfrak{AC})-$continuous. Then there exists $m_0 > \mathfrak{N}$ such that $d_{\mathfrak{Y}}(f_{\mathfrak{N}}(\mathfrak{x}_{<n,m>}), f_{\mathfrak{N}}(\mathfrak{x}_n)) \ll \frac{u'}{3}$, $\forall\ \mathfrak{x} \in \mathfrak{X}$ and $m, n \in \mathbb{Z}$ such that $< m, n > \geq m_0$. Hence we have

$$\begin{aligned} & d_{\mathfrak{Y}}(f(\mathfrak{x}_{<n,m>}), f(\mathfrak{x}_n)) \\ \leq\ & d_{\mathfrak{Y}}(f(\mathfrak{x}_{<n,m>}), f_{\mathfrak{N}}(\mathfrak{x}_{<n,m>})) + d_{\mathfrak{Y}}(f_{\mathfrak{N}}(\mathfrak{x}_{<n,m>}), f_{\mathfrak{N}}(\mathfrak{x}_n)) + d_{\mathfrak{Y}}(f_{\mathfrak{N}}(\mathfrak{x}_n), f(\mathfrak{x}_n)) \\ \ll\ & \frac{\mathfrak{u}'}{3} + \frac{u'}{3} + \frac{u'}{3} \\ =\ & u'. \end{aligned}$$

Hence the proof.

Theorem 3.12. *Let forward and backward convergence in $\mathfrak{Y}$ are equivalent. Then the set of all forward $(c\mathfrak{AC})-$continuous functions from $\mathfrak{X}$ to $\mathfrak{Y}$ is a closed subset of set of all functions from $\mathfrak{X}$ to $\mathfrak{Y}$ that are continuous.*
Proof. The proof is omitted as it follows trivially from Theorem 3.11.

Theorem 3.13. *Composition of two forward arithmetic ff-continuous functions in a quasi cone metric space $(\mathfrak{X}, d)$ is arithmetic ff-continuous.*
Proof. Let g, h be two forward arithmetic ff-continuous functions. We shall show that $g \circ h$ is again an arithmetic ff-continuous function. For this, let $\{\mathfrak{x}_n\}$ be a forward arithmetic convergent sequence in $\mathfrak{X}$. Since h is arithmetic continuous. Then $\{h(\mathfrak{x}_n)\}$ is forward arithmetic convergent. Moreover, g is also arithmetic ff-continuous. Then $\{g(h(\mathfrak{x}_n))\}$ is also forward arithmetic convergent. Hence $g \circ h$ is also arithmetic ff-continuous. This proves the theorem.

Definition 3.14. *A set $\mathfrak{B} \subset \mathfrak{X}$ is said to be forward (resp. backward) arithmetic compact if every sequence in $\mathfrak{B}$ admits forward (resp. backward) arithmetic convergent subsequence.*

Theorem 3.15. *Let $\mathfrak{B} \subset \mathfrak{X}$ be forward arithmetic compact and $f : \mathfrak{X} \to \mathfrak{Y}$ be an arithmetic $ff-$continuous function. Then $f(\mathfrak{B})$ is also forward arithmetic compact.*
Proof. Let $\{\mathfrak{y}_n\}$ be a sequence in $f(\mathfrak{B})$. Then $\mathfrak{y}_n = f(\mathfrak{x}_n)$ for some $\mathfrak{x}_n \in \mathfrak{X}$ and $n \in \mathbb{N}$. Also, $\{\mathfrak{x}_n\}$ has a forward arithmetic convergent subsequence $\{\mathfrak{x}_{n_l}\}$. By the

arithmetic $ff-$continuity of f we see that $f(\mathfrak{x}_n)$ has a forward arithmetic convergent subsequence $f(\mathfrak{x}_{n_l})$. This proves that $f(\mathfrak{B})$ is forward arithmetic compact.

Theorem 3.16. *Let $\mathfrak{B} \subset \mathfrak{X}$ be backward arithmetic compact and $f : \mathfrak{X} \to \mathfrak{Y}$ be an arithmetic $fb-$continuous function. Then $f(\mathfrak{B})$ is also backward arithmetic compact.*

Proof. The proof is omitted as it follows trivially from Theorem 3.15.

Theorem 3.17. *Let $\mathfrak{B} \subset \mathfrak{X}$ be forward arithmetic compact. Then any closed subset of $\mathfrak{B}$ is forward arithmetic compact.*

Proof. Let $\mathfrak{B} \subset \mathfrak{X}$ be forward arithmetic compact and $\mathfrak{A}$ be a closed set contained in $\mathfrak{B}$. Consider a sequence $\{\mathfrak{x}_n\}$ in $\mathfrak{A}$. Then $\{\mathfrak{x}_n\}$ is a sequence in $\mathfrak{B}$. Also, the sequence $\{\mathfrak{x}_n\}$ has a forward arithmetic convergent subsequence $\{\mathfrak{x}_{n_l}\}$ as $\mathfrak{B}$ is forward arithmetic compact. Moreover, $\mathfrak{A}$ is closed. Then the sequence $\{\mathfrak{x}_n\}$ in $\mathfrak{A}$ has a forward arithmetic convergent subsequence in $\mathfrak{A}$. Hence the proof.

Theorem 3.18. *Let $\mathfrak{B} \subset \mathfrak{X}$ be backward arithmetic compact. Then any closed subset of $\mathfrak{B}$ is backward arithmetic compact.*

Proof. The proof is omitted as it follows trivially from Theorem 3.17.

4. Conclusion

In this paper, a new notion of arithmetic continuity in quasi cone metric spaces is introduced. The notions of uniform continuity, forward $(c\mathfrak{A}\mathfrak{C})-$continuous functions, arithmetic ff-continuity, arithmetic fb-continuity, forward and backward arithmetic compactness are also presented. We established that a uniformly continuous function is arithmetic ff-continuous in the context of quasi cone metric spaces. A result pertaining the uniform limit of a sequence of arithmetic ff-continuous functions is investigated. Furthermore, certain characteristics of quasi cone metric spaces are deduced concerning their forward and backward arithmetic compactness. In this space, we shall investigate several new topological and algebraic properties.

References

[1] Abdeljawad T. and Karapinar E., Quasicone metric spaces and generalizations of Caristi Kirk's Theorem, Fixed Point Theory and Applications, 2009(1) (2009), 9-18.

[2] Albert G. E., A note on quasi-metric spaces, Bulletin of the American Mathematical Society, 47 (1941), 479-482.

[3] Cakalli H., A variation on arithmetic continuity, Boletim Da Sociedade Paranaense De Matemática, 37(2) (2019), 179-181.

[4] Chenchiah I., Rieger M. O. and Zimmer J., Gradient flows in asymmetric metric spaces, Nonlinear Analysis: Theory, Methods and Applications, 71 (2009), 5820–5834.

[5] Collins J., Zimmer J., An asymmetric Arzela-Ascoli theorem, Topology and its Applications, 154 (2007), 2312-2322.

[6] Demirci I. A., Kisi O., Gurdal M., Rough I_2-statistical convergence in cone metric spaces in certain details, Bulletin of Mathematical Analysis and Applications, 15(1) (2023), 7-23.

[7] Demirci I. A., Kisi O., Gurdal M., On quasi I-statistical convergence of triple sequence in cone metric spaces, Mathematica Pannonica, 29(1) (2023), 87-95.

[8] Gurdal M., Certain aspects of I-lacunary arithmetic statistical convergence, Journal of Applied and Pure Mathematics, 5(3) (2023), 265-279.

[9] Huang L. G., Zhang X., Cone metric spaces and fixed point theorems of contractive mappings, Journal of Mathematical Analysis and Applications, 332 (2007), 1468-1476.

[10] Huban M. B., Gurdal M., On invariant arithmetic statistically convergence via weighted density, Ilirias Journal of Mathematics, 9(1) (2021), 23-34.

[11] Kisi O., Gurdal M., New results on lacunary ideal convergence in fuzzy cone normed spaces, Sigma Journal of Engineering and Natural Sciences, 41(6) (2023), 1255-1263.

[12] Kunzi H. P. A., A note on sequentially compact quasipseudometric spaces. Monatshefte für Mathematik, 95 (1983), 219-220.

[13] Mennucci A. C. G., On asymmetric distances. Analysis and Geometry in Metric Spaces, 1 (2013), 200–231.

[14] Ruckle W. H., Arithmetical summability, Journal of Mathematical Analysis and Applications, 396 (2012), 741-748.

[15] Shadad F., Noorani M. S. Md., Fixed point results in quasi-cone metric spaces, Abstract and Applied Analysis, 2013 (2013), 7 pages.

[16] Wilson W. A., On semi-metric spaces, American Journal of Mathematics, 53 (1931), 361-373.

[17] Yaying T. and Hazarika B., On arithmetical summability and multiplier sequences, National Academy Science Letters, 40(1) (2017), 43-46.

[18] Yaying T. and Hazarika B., On arithmetic continuity, Boletim Da Sociedade Paranaense De Matemática, 35(1) (2017), 139-145.

[19] Yaying T. and Hazarika B. and Cakalli H., New results in quasi cone metric spaces, Journal of Mathematics and Computer Science, 16(3) (2016), 435-444.

[20] Yaying T., Hazarika B. and Mohiuddine S. A., Some new types of continuity in asymmetric metric spaces, Facta Universitatis Series Mathematics and Informatics, 35(2) (2020), 485-493.

[21] Yaying T. and Hazarika B., On arithmetic continuity in metric spaces, Afrika Matematica, 28 (2017), 985-989.

[22] Yaying T. and Hazarika B., Lacunary arithmetic convergence, Proceedings of the Jangjeon Mathematical Society, 21(3) (2018), 507-513.

This page intentionally left blank.